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ON CONSTRUCTIONS OF MINIMAL SURFACES

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Abstract. In the recent papers, Sánchez-Reyes [Appl. Math. Model. 40 (2016), 1676–1682] described the method for finding a minimal surface through a geodesic, and Li et al. [Appl. Math. Model. 37 (2013), 6415–6424] studied the approximation of minimal surfaces with a geodesic from Dirichlet function. In the present article, we consider an isoparametric surface generated by Frenet frame of a curve introduced by Wang et al. [Comput. Aided Des. 36 (2004), 447-459], and give the necessary and sufficient condition to satisfy both geodesic of the curve and minimality of the surface. From this, we construct minimal surfaces in terms of constant curvature and torsion of the curve. As a result, we present a new approach for constructions of the minimal surfaces from a prescribed closed geodesic and unclosed geodesic, and show some new examples of minimal surfaces with a circle and a helix as a geodesic. Our approach can be used in design of minimal surfaces from geodesics.

1. Introduction

Geometers have been interested in studying minimal surfaces and geodesics for a long times. A minimal surface is a surface with vanishing mean curvature. As the mean curvature is the variation of area functional of a surface, and minimal surfaces include the surfaces minimizing the area with a fixed boundary. Because of their appealing properties, minimal surfaces have been spiritedly studied in many research areas. In mathematics, the surfaces have wide applications in a surface design [4, 7, 9, 10, 11]. In physics, minimal surfaces are familiar as soap films. Besides the obvious application of a minimal surface theory to the study

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of soap films, there are a number of other physical systems in which the theory of minimal surfaces has a sometimes surprising applicability.

On the other hand, geodesics are curves on surfaces that plays a role analogous to straight lines in the plane. It is well known that great circles are geodesics on a sphere, and parallels (circles) and helices are geodesics on a circular cylinder. But ordinary helices on a helicoid are not geodesics, and parallel curves (circles) on a surface of revolution are not also always geodesics. Based on these facts, Tamura [12, 13] considered a helical geodesic on a surface, and he showed that complete surfaces of a constant mean curvature on which there exist two helical geodesics through each point are planes, spheres or circular cylinders. The above mentioned statement is an important role for our results in the paper, because we suggest minimal surfaces contained circles or helices as a geodesic. As applications for geodesics, Munchmeyer and Haw [8] were the first to introduce the geodesic to the CAGD community. They applied the geodesic in ship design, namely to find out the precise layout of the seams and butts in the ship hull. Also, in [3] Haw was first defined an operative sail shape by using patched parametric surface and gave the method for a sail design with a geodesic.

The study of combining minimal surfaces and geodesics appear attractive and are used many areas. In [7] Li, Wang and Zhu are mentioned that a geodesic is an important curve in a practical application, especially in shoe design and garment design, and they gave examples for approximation minimal surfaces with a geodesic by using Dirichlet function. Also, in [10] Sánchez-Reyes justified why minimal surfaces and the problem of finding the surface with minimal area have little to do with garment design. And he examined construction method of a minimal surface from a prescribed geodesic and drew minimal surfaces with circle or helix as a geodesic. Moreover, Sánchez-Reyes and Dorado [11] presented a practical method to construct polynomial surfaces from a polynomial geodesic by prescribing tangent ribbons, and Riverros and Corro [9] classified GICM-surfaces, defined by the class of minimal surfaces with an isothermal coordinate and a family of geodesic coordinate curves. Several mathematicians are studying minimal surfaces passing through geodesics [5, 6, 7, 9, 10, 16], etc. Also, geodesics and minimal surfaces are widely used for medical image segmentation [1, 2].

The paper is arranged as follows. In Section 2, after briefly reviewing some fundamental concepts of a parametric surface worked by [14] we give minimal conditions for an isoparametric minimal surface in terms of the marching-scale functions. As a result, in Section 3 we present a new approach for minimal surfaces from a given curve, in particular, a circle and a helix, and give some examples. In Section 4 we explain how to construct minimal surfaces passing through a geodesic, and give new examples for minimal surfaces.

2. Conditions of minimal surfaces

Let γ be a curve parametrized by arc-length s in Euclidean 3-space $\mathbb{E}^3$. Denote by $\{T, N, B\}$ the Frenet frame of a curve γ and κ, τ the curvature and the torsion of γ, respectively.

Consider a parametric surface generated by the curve γ and its Frenet frame as following

$$X(s,t) = \gamma(s) + (f(s,t)\ g(s,t)\ h(s,t)) \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{2.1}$$
$$s_1 \leq s \leq s_2, \quad t_1 \leq t \leq t_2,$$

where $f(s,t), g(s,t)$ and $h(s,t)$ are smooth functions.
If the parameter t is seen as the time, $f(s,t), g(s,t)$ and $h(s,t)$ can be viewed as directed marching distances of a point unit at the time t in the directions $T(s)$, $N(s)$ and $B(s)$, respectively. Sometimes, $f(s,t), g(s,t)$ and $h(s,t)$ are said to be the marching-scale functions in the directions $T(s)$, $N(s)$ and $B(s)$, respectively. Some known simple examples are to be mentioned, namely

1. If we take the marching-scale functions as $f(s,t) = g(s,t) = h(s,t) = t$, then the parametric surface $X(s,t)$ is a ruled surface.
2. If γ is a circle and $f(s,t) = 0, g(s,t) = \tilde{g}(t), h(s,t) = \tilde{h}(t)$, then the parametric surface $X(s,t)$ is a usual surface of revolution.
3. If the marching-scale functions are given by
$$\begin{aligned} f(s,t) &= -r(s)r'(s), \\ g(s,t) &= -r(s)\sqrt{1-r'(s)}\cos t, \\ h(s,t) &= r(s)\sqrt{1-r'(s)}\sin t \end{aligned}$$
with a smooth function $r(s)$, then the surface is a canal surface. In particular, if r is constant, the surface is a tubular surface.

In [14], authors defined an isogeodesic to construct a family of surfaces from a given spatial geodesic curve.

Definition 2.1. A curve $\gamma(s)$ on the parametric surface $X(s,t)$ defined by (2.1) is called an isoparametric curve if there exists a time t_0 such that $\gamma(s) = X(s,t_0)$.

Definition 2.2. A curve $\gamma(s)$ on the surface $X(s,t)$ is said to be isogeodesic of $X(s,t)$ if it is both an isoparametric curve and a geodesic on $X(s,t)$.

For the better analysis of a parametric surface, we now consider the marching-scale functions $f(s,t)$, $g(s,t)$ and $h(s,t)$ are expressed by

$$f(s,t) = u(t), \quad g(s,t) = v(t), \quad h(s,t) = w(t), \tag{2.2}$$

where $u(t)$, $v(t)$, $w(t)$ are smooth functions.

Lemma 2.3. ([14]) *A curve $\gamma(s)$ on the parametric surface $X(s,t)$ given by* (2.1) *with the marching-scale functions given by* (2.2) *is an isogeodesic if and only if the following conditions are satisfied*

$$\begin{aligned} &u(t_0) = v(t_0) = w(t_0) = 0, \\ &w'(t_0) \neq 0, \\ &v'(t_0) = 0, \end{aligned} \tag{2.3}$$

where the prime is derivative with respect to t.

Definition 2.4. If $X(s,t)$ satisfies $E = G$ and $F = 0$, then $X(s,t)$ is called a isothermal surface, where E, F and G denote the coefficients of the first fundamental form of a surface $X(s,t)$.

Definition 2.5. If $X(s,t)$ satisfies $\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 X}{\partial t^2} = 0$, then $X(s,t)$ is called a harmonic surface.

Definition 2.6. If $X(s,t)$ has a vanishing mean curvature, then $X(s,t)$ is called a minimal surface.

Lemma 2.7. ([15]) *The surface with an isothermal parameter is minimal if and only if it is a harmonic surface.*

Now, we give the minimal conditions of the parametric surface $X(s,t)$ with the marching-scale functions given as (2.2). Also, the following theorem is useful for our results.

Theorem 2.8. *Let γ be a unit speed isoparametric curve on a surface in Euclidean 3-space. The surface parametrized by*

$$X(s,t) = \gamma(s) + u(t)T(s) + v(t)N(s) + w(t)B(s) \tag{2.4}$$

is minimal if and only if there exist the marching-scale functions u, v *and* w *satisfying the following conditions:*

$$(1-\kappa(s)v(t))^2 + (\kappa(s)u(t) - \tau(s)w(t))^2 + \tau^2(s)v^2(t) \\ -u'^2(t) - v'^2(t) - w'^2(t) = 0. \tag{2.5}$$

$$u'(t)(1-\kappa(s)v(t)) + v'(t)(\kappa(s)u(t) - \tau(s)w(t)) + \tau(s)v(t)w'(t) = 0. \tag{2.6}$$

$$\kappa'(s)u(t) + \kappa^2(s)u(t) - \kappa(s)\tau(s)w(t) - u''(t) = 0. \tag{2.7}$$

$$\kappa'(s)u(t) - \tau'(s)w(t) + \kappa(s) - \kappa^2(s)v(t) - \tau^2(s)v(t) + v''(t) = 0. \tag{2.8}$$

$$\tau'(s)v(t) + \kappa(s)\tau(s)u(t) - \tau^2(s)w(t) + w''(t) = 0. \tag{2.9}$$

Proof. From the surface equation (2.4), we have

$$\frac{\partial X}{\partial s} = (1-\kappa(s)v(t))T(s) + (\kappa(s)u(t) - \tau(s)w(t))N(s) + \tau(s)v(t)B(s),$$
$$\frac{\partial X}{\partial t} = u'(t)T(s) + v'(t)N(s) + w'(t)B(s).$$

It follows that the coefficients of the first fundamental form of the surface are

(2.10)
$$E = (1-\kappa(s)v(t))^2 + (\kappa(s)u(t) - \tau(s)w(t))^2 + \tau^2(s)v^2(t),$$
$$F = u'(t)(1-\kappa(s)v(t)) + v'(t)(\kappa(s)u(t) - \tau(s)w(t)) + \tau(s)v(t)w'(t),$$
$$G = u'^2(t) + v'^2(t) + w'^2(t).$$

Also, the second derivatives of the surface $X(s,t)$ are given by

(2.11)
$$\frac{\partial^2 X}{\partial s^2} = \left[-\kappa'(s)u(t) - \kappa^2(s)u(t) + \kappa(s)\tau(s)w(t)\right] T(s) \\ + \left[\kappa'(s)u(t) - \tau'(s)w(t) + \kappa(s) - \kappa^2(s)v(t) - \tau^2(s)v(t)\right] N(s) \\ + \left[\tau'(s)v(t) + \kappa(s)\tau(s)u(t) - \tau^2(s)w(t)\right] B(s),$$
$$\frac{\partial^2 X}{\partial t^2} = u''(t)T(s) + v''(t)N(s) + w''(t)B(s).$$

From an isothermal condition, equation (2.10) gives (2.5) and (2.6). Also, a harmonic equation $\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 X}{\partial t^2} = 0$ implies (2.7), (2.8) and (2.9). □

In order to construct minimal surfaces, we have to solve the above system of ordinary differential equations. But, it is difficult to find exact solutions satisfying (2.5)-(2.9) for minimal surfaces. In this paper, our goal is to find a helix as a geodesic on a minimal surface. So we consider the partial solutions of minimality conditions of Theorem 2.8 in terms of constant curvature $\kappa(s)$ and torsion $\tau(s)$.

3. Constructions of minimal surfaces generated by curves

Let γ be a unit speed curve in Euclidean 3-space and X be a surface parametrized by

$$X(s,t) = \gamma(s) + u(t)T(s) + v(t)N(s) + w(t)B(s). \tag{3.1}$$

Now, in order to obtain our results we split it into two cases according to constant curvature and torsion.

3.1. γ is a circle

Suppose that a curve γ has a non-zero constant curvature $\kappa = k_0$ and a zero torsion. Then, from (2.7), (2.8) and (2.9) we can obtain

$$\begin{aligned} u(t) &= a_1 e^{k_0 t} + a_2 e^{-k_0 t}, \\ v(t) &= b_1 e^{k_0 t} + b_2 e^{-k_0 t} + \frac{1}{k_0}, \\ w(t) &= c_1 t + c_2 \end{aligned} \tag{3.2}$$

for constants a_i, b_i, c_i, $i = 1, 2$.
Substituting (3.2) into (2.5) and (2.6) we have the relationship as

$$4k_0^2(a_1 a_2 + b_1 b_2) - c_1^2 = 0, \tag{3.3}$$

$$a_1 b_2 - a_2 b_1 = 0, \tag{3.4}$$

respectively.

Since γ has a non-zero constant curvature k_0 and a zero torsion, γ is a circle. By a rigid motion, we consider γ parametrized by

$$\gamma(s) = \left(\frac{1}{k_0} \cos(k_0 s), \frac{1}{k_0} \sin(k_0 s), 0 \right). \tag{3.5}$$

Then the Frenet frame of the circle is given by

$$\begin{aligned} T(s) &= (-\sin(k_0 s), \cos(k_0 s), 0), \\ N(s) &= (-\cos(k_0 s), -\sin(k_0 s), 0), \\ B(s) &= (0, 0, 1). \end{aligned} \tag{3.6}$$

1. If $c_1 = 0$, we get $w(t) = c_2$. In this case, the isoparametric surface X can be expressed as

$$X(s,t) = \begin{pmatrix} (\frac{1}{k_0} - v(t))\cos(k_0 s) - u(t)\sin(k_0 s)) \\ (\frac{1}{k_0} - v(t))\sin(k_0 s) + u(t)\cos(k_0 s) \\ c_2 \end{pmatrix}$$

that is, the surface is a plane.

Now, we study non-planar minimal surface, that is, $c_1 \neq 0$.

2. If $b_1 = 0$, from (3.4) one find $a_1 = 0$ or $b_2 = 0$. If $a_1 = 0$, from (3.3) $c_1 = 0$, a contradiction. Thus, $b_2 = 0$. From this, equation (3.2) deduced as the form:

$$\begin{aligned} u(t) &= a_1 e^{k_0 t} + a_2 e^{-k_0 t}, \\ v(t) &= \frac{1}{k_0}, \\ w(t) &= c_1 t + c_2, \end{aligned} \tag{3.7}$$

which implies that the surface is parametrized as

$$X(s,t) = \left(-d_1\cosh(k_0 t + d_2)\sin(k_0 s), d_1\cosh(k_0 t + d_2)\cos(k_0 s), c_1 t + c_2\right),$$

where d_1, d_2 are constant. By a rigid motion, the surface is obtained by rotating the curve $y = d\cosh(k_0 z)$ in the yz-plane around the z-axis and it is a catenoid.

On the other hand, (3.3) and (3.4) are symmetric about a_1, a_2 and b_1, b_2. So, in the other case, that is, $a_1 = 0$ the surface is a catenoid.

3. If a_1, a_2, b_1, b_2 are non-zero constant, equation (3.4) implies

$$a_2 = a_1 r_0, \quad b_2 = b_1 r_0$$

for non-zero constant r_0. Also equation (3.3) becomes

$$4\kappa_0^2(r_0 a_1^2 + r_0 b_1^2) - c_1^2 = 0,$$

it follows that the constant r_0 must be positive. We put

$$q_0^2 = r_0 a_1^2 + r_0 b_1^2$$

for a non-zero constant q_0. Then we get

$$a_1 = \frac{q_0}{\sqrt{r_0}}\cos\theta_0, \quad b_1 = \frac{q_0}{\sqrt{r_0}}\sin\theta_0.$$

From this, the marching-scale functions u, v and w are written as

$$u(t) = q_0\sqrt{r_0}\cos\theta_0(e^{k_0t} + e^{-k_0t}) = 2q_0\sqrt{r_0}\cos\theta_0\cosh(k_0t),$$

$$v(t) = q_0\sqrt{r_0}\sin\theta_0(e^{k_0t} + e^{-k_0t}) + \frac{1}{k_0} = 2q_0\sqrt{r_0}\sin\theta_0\cosh(k_0t) + \frac{1}{k_0},$$

$$w(t) = 2\kappa_0q_0t + c_2.$$

If a circle γ is given by (3.5), a minimal surface $X(s,t)$ passing through $\gamma(s)$ can be expressed as the form

$$X(s,t) = \left(p_0\cosh(k_0t)\cos(s + \frac{\pi}{4}), p_0\cosh(k_0t)\sin(s + \frac{\pi}{4}), 2k_0q_0t + c_2\right),$$

where $p_0 = q_0\sqrt{r_0}\cos\theta_0$. This means that the surface is a catenoid. Thus, we have

Theorem 3.1. *Let X be a surface parametrized by*

$$X(s,t) = \gamma(s) + u(t)T(s) + v(t)N(s) + w(t)B(s). \tag{3.8}$$

If the surface X passing through a circle γ with a non-zero constant curvature k_0 is minimal, then it is either part of a plane or part of a catenoid.

Remark 3.2. It is well known that a minimal surface of revolution is either part of a plane or part of a catenoid. So, the surface $X(s,t)$ given by (3.8) passing through a circle is a surface of revolution.

3.2. γ is a helix

Suppose that a curve γ has non-zero constant curvature κ_0 and torsion τ_0. Equation (2.9) implies

$$u(t) = \frac{1}{\kappa_0\tau_0}\left(-w''(t) + \tau_0^2w(t)\right). \tag{3.9}$$

If we substitute (3.9) in (2.7), one find

$$w^{(4)}(t) - (\kappa_0^2 + \tau_0^2)w''(t) = 0,$$

its general solution is

$$w(t) = c_1e^{m_0t} + c_2e^{-m_0t} + c_3t + c_4, \tag{3.10}$$

where $m_0^2 = \kappa_0^2 + \tau_0^2$ and c_i $(i = 1, ..., 4)$ are constant. Substituting (3.10) into (3.9), a function u is given by

$$u(t) = -\frac{\kappa_0}{\tau_0}\left(c_1e^{m_0t} + c_2e^{-m_0t}\right) + c_3\frac{\tau_0}{\kappa_0}t + c_4\frac{\tau_0}{\kappa_0}. \tag{3.11}$$

Since (2.8) is the second order linear differential equation, we can solve it, and so its general solution is

$$v(t) = b_1 e^{m_0 t} + b_2 e^{-m_0 t} + \frac{\kappa_0}{m_0^2}. \tag{3.12}$$

If we substitute (3.10), (3.11) and (3.12) into (2.5) and (2.6), we can check that the coefficients of $e^{m_0 t}$ and $e^{-m_0 t}$ are zero. Thus from the constant terms of (2.5) and (2.6) we obtain

$$c_3^2 m_0^4 \tau_0^2 - 4b_1 b_2 m_0^4 \kappa_0^2 \tau_0^2 - 4c_1 c_2 m_0^6 \kappa_0^2 - \kappa_0^2 \tau_0^4 = 0, \tag{3.13}$$

$$c_3 \tau_0^2 + 2m_0^3 \kappa_0 (b_2 c_1 - c_2 b_1) = 0, \tag{3.14}$$

respectively. Thus, we have

THEOREM 3.3. *Let X be a surface parametrized by*

$$X(s,t) = \gamma(s) + u(t)T(s) + v(t)N(s) + w(t)B(s). \tag{3.15}$$

If the surface X passing through a helix γ with a non- zero constant curvature κ_0 and a torsion τ_0 in Euclidean 3-space is minimal, then the marching-scale functions u, v and w are expressed as the form:

$$\begin{aligned} u(t) &= -\frac{\kappa_0}{\tau_0}\left(c_1 e^{m_0 t} + c_2 e^{-m_0 t}\right) + c_3 \frac{\tau_0}{\kappa_0} t + c_4 \frac{\tau_0}{\kappa_0}, \\ v(t) &= b_1 e^{m_0 t} + b_2 e^{-m_0 t} + \frac{\kappa_0}{m_0^2}, \\ w(t) &= c_1 e^{m_0 t} + c_2 e^{-m_0 t} + c_3 t + c_4, \end{aligned} \tag{3.16}$$

where $m_0^2 = \kappa_0^2 + \tau_0^2$ and constants $b_1, b_2,\ c_i\ (i = 1, .., 4)$ satisfy the following

$$\begin{aligned} c_3^2 m_0^4 \tau_0^2 - 4b_1 b_2 m_0^4 \kappa_0^2 \tau_0^2 - 4c_1 c_2 m_0^6 \kappa_0^2 - \kappa_0^2 \tau_0^4 &= 0, \\ c_3 \tau_0^2 + 2m_0^3 \kappa_0 (b_2 c_1 - c_2 b_1) &= 0. \end{aligned} \tag{3.17}$$

REMARK 3.4. There are infinite numbers of minimal surfaces passing through a helix for constants $b_1, b_2,\ c_i\ (i = 1, .., 4)$.

EXAMPLE 3.5. *Consider a helix parametrized by*

$$\gamma(s) = \left(\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}} s\right). \tag{3.18}$$

Then we have the Frenet frame of the curve as

$$
\begin{aligned}
T(s) &= (-\frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}),\\
N(s) &= (-\cos s, -\sin s, 0), \qquad (3.19)\\
B(s) &= (\frac{1}{\sqrt{2}}\sin s, -\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}),
\end{aligned}
$$

and the curvature κ *and the torsion* τ *are given by* $\kappa = \frac{1}{\sqrt{2}}$ *and* $\tau = \frac{1}{\sqrt{2}}$, *respectively, it follows that* $m_0^2 = 1$. *Now, to construct a minimal surface we take*

$$b_1 = -b_2 = \frac{1}{4}, \quad c_1 = -c_2 = \frac{1}{4\sqrt{2}}, \quad c_3 = c_4 = 0.$$

In this case, the marching-scale functions u, v *and* w *are written as*

$$
\begin{aligned}
u(t) &= -\frac{1}{2\sqrt{2}}\sinh t,\\
v(t) &= \frac{1}{2}\sinh t + \frac{1}{\sqrt{2}},\\
w(t) &= \frac{1}{2\sqrt{2}}\sinh t.
\end{aligned}
$$

From this, the minimal surface X *passing through the helix* γ *can be parametrized as*

$$X(s,t) = \left(\frac{\sqrt{3}}{2}\sinh t\sin(s+\theta_0), \frac{\sqrt{3}}{2}\sinh t\cos(s+\theta_0), \frac{1}{\sqrt{2}}s\right),$$

where $\cos\theta_0 = \frac{1}{\sqrt{3}}$ *and the surface* X *is a helicoid.*

Example 3.6. *Considering a helix given by* (3.18) *and taking*

$$b_1 = b_2 = 1, \quad c_1 = \frac{3}{4\sqrt{2}}, \quad c_2 = 0, \quad c_3 = -\frac{3}{2}, \quad c_4 = 0.$$

Then, the marching-scale functions u, v *and* w *are*

$$
\begin{aligned}
u(t) &= -\frac{3}{4\sqrt{2}}e^t - \frac{3}{2}t,\\
v(t) &= 2\cosh t + \frac{1}{2},\\
w(t) &= \frac{3}{4\sqrt{2}}e^t - \frac{3}{2}t,
\end{aligned}
$$

it follows that the minimal surface X passing through the helix γ can be parametrized as

$$X(s,t) = \begin{pmatrix} -2\cosh t\cos s + \frac{3}{4}e^t \sin s \\ -2\cosh t \sin s - \frac{3}{4}e^t\cos s \\ \frac{\sqrt{2}}{2}s - \frac{3\sqrt{2}}{2}t \end{pmatrix}$$

4. Representation of minimal surfaces with a geodesic

In this section, we consider a minimal surface parametrized by

$$X(s,t) = \gamma(s) + u(t)T(s) + v(t)N(s) + w(t)B(s) \tag{4.1}$$

passing through a curve γ, and in particular construct the minimal surfaces passing through a circle (closed geodesic) and a helix (unclosed geodesic).

4.1. Minimal surfaces with a circle as a closed geodesic

Let X be a minimal surface passing through a circle γ. Suppose that a circle $\gamma(s)$ on a surface $X(s,t)$ is a geodesic. Then the geodesic condition (2.3) with the help of (3.2) implies

$$\begin{aligned} &a_2 = -a_1 e^{2\kappa_0 t_0}, \\ &b_1 = -\frac{1}{2\kappa_0}e^{-\kappa_0 t_0}, \quad b_2 = -\frac{1}{2\kappa_0}e^{\kappa_0 t_0}, \\ &c_1 \neq 0, \quad c_2 = -c_1 t_0. \end{aligned} \tag{4.2}$$

If we substitute (4.2) into (3.4), we can obtain $a_1 = 0$ and $a_2 = 0$, that is, a function u is identically zero. Also, combining (3.3) and (4.2) one find $c_1 = \pm 1$. Thus, (3.2) reduces to

$$\begin{aligned} u(t) &= 0, \\ v(t) &= -\frac{1}{2\kappa_0}\left(e^{-\kappa_0 t_0}e^{\kappa_0 t} + e^{\kappa_0 t_0}e^{-\kappa_0 t}\right) + \frac{1}{\kappa_0}, \\ w(t) &= \pm t \mp t_0. \end{aligned}$$

Thus, we have

THEOREM 4.1. *Let X be a surface parametrized by*

$$X(s,t) = \gamma(s) + u(t)T(s) + v(t)N(s) + w(t)B(s). \tag{4.3}$$

If the surface X passing through a circle γ with a non- zero constant curvature κ_0 in Euclidean 3-space is minimal and the circle γ is a geodesic on the minimal surface at $t = t_0$, then the marching-scale functions u, v and w are expressed as the form:

$$\begin{aligned} u(t) &= 0, \\ v(t) &= -\frac{1}{2\kappa_0}\left(e^{-\kappa_0 t_0}e^{\kappa_0 t} + e^{\kappa_0 t_0}e^{-\kappa_0 t}\right) + \frac{1}{\kappa_0}, \\ w(t) &= \pm t \mp t_0. \end{aligned}$$

EXAMPLE 4.2. *Consider a circle γ with radius 1. Then for $t_0 = 0$ the minimal surface $X(s,t)$ passing through the circle as a geodesic is parametrized as*

$$X(s,t) = (\cosh t \cos s, \cosh t \sin s, t).$$

REMARK 4.3. (1) The surface in Example 4.2 is a catenoid. It is well-known that a parallel $t = t_0$ on the catenoid is a geodesic if and only if $\frac{df}{dt} = 0$ when $t = t_0$, that is, t_0 is a stationary point of f, where $f(t) = \cosh t$. Thus a parallel $t_0 = 0$ is a geodesic on the catenoid.

(2) Usually, a catenoid is obtained by rotating the curve $x = \cosh z$ in the xz-plane around the z-axis and is one of surfaces of revolution. As a new approach of the catenoid, the surface can be explained as the form:

$$X(s,t) = \gamma(s) + v(t)N(s) + w(t)B(s),$$

where γ is a circle with radius 1 and N, B are the principal normal vector and the binormal vector of the circle, respectively. Here the functions v and w are given by $v(t) = -\cosh t + 1$ and $w(t) = t$.

4.2. Minimal surfaces with a helix as a unclosed geodesic

If a curve γ is a geodesic on the minimal surface passing through a helix, then the geodesic condition (2.3) with the help of (4.6) implies

$$\begin{aligned} \frac{\kappa_0}{\tau_0}\left(c_1 e^{m_0 t_0} + c_2 e^{-m_0 t_0}\right) - c_3\frac{\tau_0}{\kappa_0}t_0 - c_4\frac{\tau_0}{\kappa_0} &= 0, \\ b_1 e^{m_0 t_0} + b_2 e^{-m_0 t_0} + \frac{\kappa_0}{m_0^2} &= 0, \\ c_1 e^{m_0 t_0} + c_2 e^{-m_0 t_0} + c_3 t_0 + c_4 &= 0, \\ b_1 e^{m_0 t_0} - b_2 e^{-m_0 t_0} &= 0, \\ c_1 m_0 e^{m_0 t_0} - c_2 m_0 e^{-m_0 t_0} + c_3 &\neq 0. \end{aligned} \tag{4.4}$$

The first and the third equations in (4.4) imply $c_3t_0 + c_4 = 0$, and the second and the forth equations give

$$b_1 = -\frac{\kappa_0}{2m_0^2}e^{-m_0t_0}, \quad b_2 = -\frac{\kappa_0}{2m_0^2}e^{m_0t_0}.$$

Since $c_3t_0 + c_4 = 0$, from the third equation in (4.4) we get

$$c_2 = -e^{2m_0t_0}c_1.$$

Using the above data, the second equation gives

$$c_3 = \frac{2m_0\kappa_0^2}{\tau_0^2}e^{m_0t_0}c_1,$$

and the first equation with the help of constants b_1, b_2, c_3 implies

$$c_1 = \pm\frac{\tau_0^2}{2m_0^3}e^{-m_0t_0}.$$

Consequently, we have the following result.

THEOREM 4.4. *Let X be a surface parametrized by*

$$X(s,t) = \gamma(s) + u(t)T(s) + v(t)N(s) + w(t)B(s). \tag{4.5}$$

If the surface X passing through a helix γ with a non- zero constant curvature κ_0 and torsion τ_0 in Euclidean 3-space is minimal and the helix γ is a geodesic on the minimal surface at $t = t_0$, then the marching-scale functions u, v and w are expressed as the form:

(4.6)

$$\begin{aligned}
u(t) &= -\frac{\kappa_0}{\tau_0}\left(\pm\frac{\tau_0^2}{2m_0^3}e^{-m_0t_0}e^{m_0t} \mp \frac{\tau_0^2}{2m_0^3}e^{m_0t_0}e^{-m_0t}\right) \pm \frac{\kappa_0\tau_0}{m_0^2}t \mp \frac{\kappa_0\tau_0}{m_0^2}t_0,\\
v(t) &= -\frac{\kappa_0}{2m_0^2}\left(e^{-m_0t_0}e^{m_0t} + e^{m_0t_0}e^{-m_0t}\right) + \frac{\kappa_0}{m_0^2},\\
w(t) &= \pm\frac{\tau_0^2}{2m_0^3}e^{-m_0t_0}e^{m_0t} \mp \frac{\tau_0^2}{2m_0^3}e^{m_0t_0}e^{-m_0t} \pm \frac{\kappa_0^2}{m_0^2}t \mp \frac{\kappa_0^2}{m_0^2}t_0.
\end{aligned}$$

EXAMPLE 4.5. *Consider a helix parametrized by* (3.18). *If the curve is a geodesic on the minimal surface $X(s,t)$ passing through the helix γ, then in this case, the marching-scale functions are given by*

$$\begin{aligned}
u(t) &= -\frac{1}{2}(\sinh t - t),\\
v(t) &= -\frac{1}{\sqrt{2}}(\cosh t - 1),\\
w(t) &= \frac{1}{2}(\sinh t + t).
\end{aligned}$$

It follows that the minimal surface $X(s,t)$ passing through the helix as a geodesic is parametrized as

$$X(s,t) = \begin{pmatrix} \frac{1}{\sqrt{2}}\cosh t \cos s + \frac{1}{\sqrt{2}}\sinh t \sin s \\ \frac{1}{\sqrt{2}}\cosh t \sin s - \frac{1}{\sqrt{2}}\sinh t \cos s \\ \frac{1}{\sqrt{2}}s + \frac{1}{\sqrt{2}}t \end{pmatrix}$$

REMARK 4.6. It is well known that a helix on a circular cylinder is a geodesic. But a circular cylinder is not minimal. Also, a helicoid is a minimal surface and a helix is not geodesic on a helicoid. According to our approach, one can find a helix as a geodesic on a minimal surface.

5. Conclusions

Geodesics and minimal surfaces are very interesting topics in differential geometry and have a wide area of applications in natural sciences. Geodesics are used in sail design, shoe design and clothing design, etc and minimal surfaces are familiar as soap films. Also, combination of geodesics and minimal surfaces, that is, a minimal surface through prescribed geodesic is studied by Sánchez-Reyes [10].

If we use Wang's[14] isogeodesic parameter surface because of the Frenet frame of a curve, we describe the minimality condition of the isogeodesic parameter surface. As a result, we give rise to the minimal surfaces in terms of the marching-scale functions of the surface. Finally we construct minimal surfaces passing through closed geodesic(circle) and unclosed geodesic(helix).

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